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A HAMILTONIAN DECOMPOSITION OF DEGREE FOUR CAYLEY GRAPHS ON ABELIAN GROUPS

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Abstract - In a communications network, hamiltonian cycles are important because they provide a routing which can tolerate an edge failure. On the other hand, the problem of finding a hamiltonian cycle for a general graph is known to be NP-complete. This paper gives actual algorithms to decompose into two hamiltonian cycles, classes of degree four Cayley graphs on abelian groups.

INTRODUCTION

Let G be any finite abelian group and S a generating set of G that does not contain the identity element. The Cayley graph $Cay\,(S:G)$ is defined to be the graph with vertex set G and edge set $\{(g, gs) \mid g \in G,\ s \in S \cup S^{-1}\}$. Thus $Cay\,(S:G)$ is a simple connected graph that is regular of degree $|S \cup S^{-1}|$.

Given a simple graph, a hamiltonian cycle is a sequence of edges that starts at a vertex and passes through each vertex exactly once before ending at the initial vertex. A $2k-$ regular graph Γ is said to have a hamiltonian decomposition if the edges of Γ can be partitioned into k hamiltonian cycles.

Alspach made the following conjecture in 1984 (Alspach 1984):
Every $2k-$ regular Cayley graph on an abelian group has a hamiltonian decomposition.

Though the conjecture has been verified to be true in various cases, it has remained a conjecture. (See Curran & Gallian 1996 and Alspach et al. 1990 for a survey of some related results.) When the degree is four, Bermond et al. (1989) showed using a recursive method that any Cayley graph on an abelian group is decomposable into two hamiltonian cycles. Following the approach used by Bermond et al., we present here algorithms to actually construct a hamiltonian decomposition of certain classes of degree four Cayley graph on abelian groups.

The Degree Four Cayley Graph on an Abelian Group

Let G be a finite abelian group and S a generating set of G that does not contain the identity element. Suppose that the Cayley graph $Cay\,(S:G)$ has degree four. Then $|S \cup S^{-1}| = 4$ and one of the following cases must hold:

1. S consists of four elements that have order 2.
2. Exactly two elements in $S \cup S^{-1}$ have order 2.
3. $S \cup S^{-1}$ consists of two elements with order at least 3 and their inverses.

It can be shown that under the first case, G must be either $(\mathbb{Z}_2)^3$ or $(\mathbb{Z}_2)^4$. We can then easily construct a hamiltonian decomposition of the resulting Cayley graphs.

For the second case, $G \cong \mathbb{Z}_4 \times \mathbb{Z}_m$ where m is the order of the third element in S. ($\mathbb{Z}_4$ is formed by the elements of S with order 2.) A decomposition into hamiltonian cycles can be obtained by following the construction in the next section (the case G is a direct product of $\langle a \rangle$ and $\langle b \rangle$).

To deal with the third case, we shall first explain some notations and mention results to simplify the discussion.

From hereon, suppose $S = \{a, b\}$ with both a and b of order at least 3 (and $a^{-1} \neq b$). Let $\alpha = [G : \langle a\rangle]$, $\beta = [G : \langle b\rangle]$ and $k = o(a)$ (the order of a). Then $k \geq 3$, $\alpha \geq 1$ and $\beta \geq 1$. Since $\langle a, b\rangle = G$, it follows that $ab \in \langle a\rangle$ and so there exists an integer c, $0 \leq c \leq k-1$, such that $ab = ca$. It can then be verified that $\beta = \gcd(c, k)$.

As an example, we can take $G = \mathbb{Z}_{12}$, $S = \{a, b\}$, with $a = 3$ and $b = 2$. Then $k = 4$, $\alpha = 3$, $\beta = 2$ and $c = 2$. The resulting Cayley graph is shown in Fig. 1.

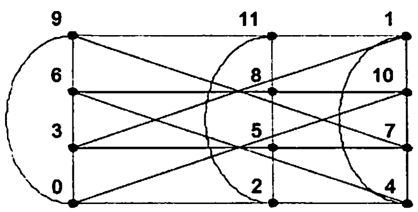

Figure1. $Cay(\{3, 2\} : \mathbb{Z}_{12})$.

In general, since G is the disjoint union of the distinct cosets of $\langle a\rangle$ in G, or

$$G = \dot{\bigcup_{0 \leq i \leq \alpha - 1}} (\langle ib\rangle + \langle a\rangle),$$

it follows that the elements of G can be represented uniquely by

$$ib + ja, \text{ or simply } (i, j),$$

where $0 \leq i \leq \alpha - 1$ and $0 \leq j \leq k-1$. The edges then consist of the following:

$\{((i, j), (i, j+1)) \mid 0 \leq i \leq \alpha - 1, 0 \leq j \leq k-1\}$,
$\{((i, j), (i+1, j)) \mid 0 \leq i \leq \alpha - 2, 0 \leq j \leq k-1\}$,
and $\{((\alpha - 1, j), (0, j + c)) \mid 0 \leq j \leq k-1\}$

(where addition in the second component is done modulo k). Figure 2 shows the same Cayley graph $Cay(\{3, 2\} : \mathbb{Z}_{12})$ with the vertices labeled using the ordered pair representation (i, j) described above.

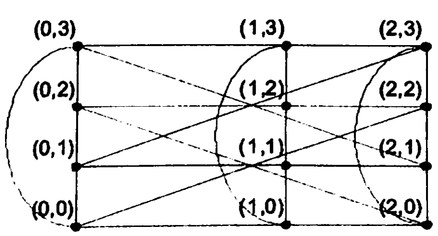

Figure 2. $Cay(\{3, 2\} : \mathbb{Z}_{12})$.

In general, a Cayley graph $Cay(S : G)$ with parameters α and β as defined above, shall be called a graph of type $\Gamma(\alpha, \beta)$. Thus, any Cayley graph belonging to Case 3 is a graph of type $\Gamma(\alpha, \beta)$ for some parameters α, β, c and k. In fact, the converse is also true. That is, given integers α, β, c and k that satisfy the conditions

$$\alpha \geq 1, \beta \geq 1, 0 \leq c \leq k-1 \text{ with } \gcd(c, k) = \beta, k \geq 3$$

then the degree-4 graph with vertices (i, j) with $0 \leq i \leq \alpha - 1$ and $0 \leq j \leq k-1$ and edges enumerated above is a Cayley graph $Cay(S : G)$ where S consists of two elements with order at least 3. (Bermond et al. 1989).

Given $Cay(\{a, b\} : G)$ and $g \in G$, we shall use the following notation:

$$g, +b, 2 * [+a, 2 * (+b)], 3 * (-a), 2 * (-b), +a$$

to denote the path that starts from g and goes through other vertices as shown in Fig. 3.

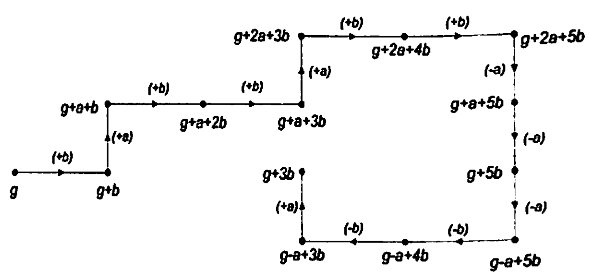

Figure 3. The path $g, +b, 2 * [+a, 2 * (+b)], 3 * (-a), 2 * (-b), +a$.

Most of the algorithms described in the next sections were obtained by applying on smaller graphs, the following two results.

Lemma 1: Let Γ be a four regular Cayley graph of type $\Gamma(\alpha, \beta)$ that has a hamiltonian decomposition $\{H_1, H_2\}$ such that both H_1 and H_2 have an edge of the form $\{(\alpha - 1, j), (0, j + c)\}$. For any integer $r \geq 1$, let Γ' be the Cayley graph of type $\Gamma(\alpha + 2r, \beta)$ that has the same parameters c, β and k (as the graph Γ). Suppose further that $\Gamma' = Cay(S' : G')$ where $S' = \{a', b'\}$ such that $o(a') = k$, $\alpha + 2r = [G' : \langle a'\rangle]$ and $(\alpha + 2r) b' = ca'$. Let H_1' (respectively, H_2') be the subgraph of Γ' obtained from H_1 (respectively, H_2) by removing all edges of the form $\{(\alpha - 1, j), (0, j + c)\}$ and replacing each one with the path

$$(\alpha - 1, j), r * [+b', (\ell - 1) * (+a'), +b', (\ell - 1) * (-a')], +b',$$

where ℓ is the smallest integer, $1 \leq \ell \leq k$, for which the edge $((\alpha - 1, j + \ell), (0, j + \ell + c))$ is in H_1(respectively, H_2). Then H_1' and H_2' form a hamiltonian decomposition of Γ'.

Lemma 2: Let $\Gamma = Cay(S : G)$ be a four regular Cayley graph of type $\Gamma(\alpha, \beta)$ with $S = \{a, b\}$ and parameters $1 \leq \alpha \leq \beta \leq 2$, c and k. Let H_1 be the subgraph obtained from the one or two cycles generated by a by removing the edges $(0, a)$ and $(-b, a - b)$ and adding $(0, -b)$ and $(a, a - b)$, and $H_2 = \Gamma \setminus E(H_1)$, the subgraph obtained by removing all edges in H_1 from Γ. Then H_1 and H_2 form a hamiltonian decomposition of Γ.

We are now ready to describe the algorithms to obtain a hamiltonian decomposition of degree four Cayley graphs of type $\Gamma(\alpha, \beta)$. We divide the graphs into two classes according to whether or not G is the direct product of $\langle a\rangle$ and $\langle b\rangle$.

The case when G is the direct product of $\langle a\rangle$ and $\langle b\rangle$

Let $Cay(S : G)$ be of type $\Gamma(\alpha, \beta)$ where $S = \{a, b\}$, $o(a) = k \geq 3$, $\alpha = [G : \langle a\rangle]$, $\beta = [G : \langle b\rangle]$ and $ab = ca$, $0 \leq c \leq k-1$. Moreover, suppose $G = \langle a\rangle \oplus \langle b\rangle$. Then $o(b) = \alpha \geq 3$, $k = \beta \geq 3$ and $c = 0$. An example is when $G = \mathbb{Z}_6 \oplus \mathbb{Z}_4$, $a = (0, 1)$ and $b = (1, 0)$. Here, $\alpha = 6$, and $\beta = k = 4$. Notice that the (i, j) notation described above coincides with the usual representation of the elements in G. The resulting Cayley graph is shown in Fig. 4. A second example is $G = \mathbb{Z}_{30}$ with $a = 5$ and $b = 12$. This time, $\alpha = 5$, and $\beta = k = 6$ and the graph is shown in Fig. 5.

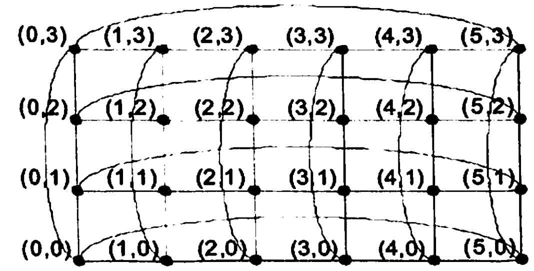

Figure 4. $Cay(\{(0, 1), (1, 0)\} : \mathbb{Z}_6 \oplus \mathbb{Z}_4)$.

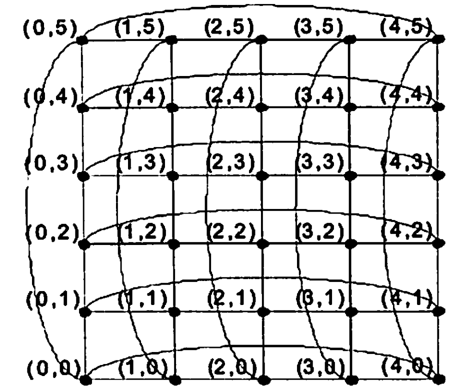

Figure 5. $Cay(\{5, 12\} : \mathbb{Z}_{30})$.

Proposition 1: Let Γ be a degree four Cayley graph $Cay(S : G)$ of type $\Gamma(\alpha, \beta)$ with parameters a, b, α, β, c and k such that $G = \langle a\rangle \oplus \langle b\rangle$ (or equivalently, $c = 0$).

(a). If α and β are both even or both odd and $\alpha \geq \beta$, let H_1 be the subgraph of Γ defined by the following sequence:

$$0, \beta * [(\beta - 1) * (+a), +b], \left(\frac{\alpha - \beta}{2}\right) * [(\beta - 1) * (+a), +b, (\beta - 1) * (-a), +b].$$

(b). If $\alpha = 2s + 1$ is odd and $\beta = 2t$ is even, let H_1 be the subgraph of Γ defined by the following sequence:

$$0, (2s) * (-b), +a, 2 * (-b), (s - 1) * [2 * (+a), -b, 2 * (-a), -b],$$

$$2 * (+a), -b, -a, -b, 2 * (+a), (t - 2) * [(2s) * (+b), +a, (2s) * (-b), +a]$$

In either case, let $H_2 = \Gamma \setminus E(H_1)$. Then H_1 and H_2 form a hamiltonian decomposition for Γ.

Proposition 1(a) can be proved by noting that the decomposition works in the case $\alpha = \beta$ and then extending this decomposition of the graph of type $\Gamma(\beta, \beta)$ to any graph of type $\Gamma(\alpha, \beta)$ by Lemma 1. Proposition 1(b) can be proved in a similar way. We note that Proposition 1 can be applied also if either $\alpha < \beta$ in (a) or if α is even while β is odd in (b), by first interchanging the values of α and β. Illustrations are given in Fig. 6 and Fig. 7.

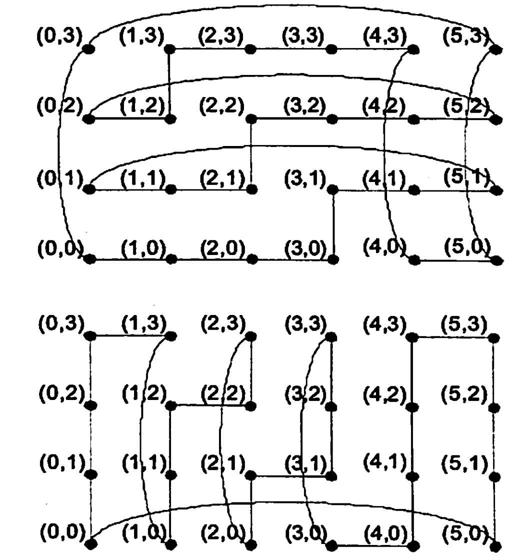

Figure 6. Hamiltonian decomposition of $Cay(\{(0,1),(1,0)\} : \mathbb{Z}_6 \oplus \mathbb{Z}_4)$.

The case when G is *not* the direct product of $\langle a \rangle$ and $\langle b \rangle$

As in the previous section, we take G to be any finite abelian group generated by $S = \{a, b\}$ with both a and b of order at least 3 (and $a^{-1} \neq b$), $\alpha = [G : \langle a \rangle]$, $\beta = [G : \langle b \rangle]$, $k = o(a)$ and $0 \leq c \leq k - 1$ with $\alpha b = ca$. This time, we assume $G \neq \langle a \rangle \oplus \langle b \rangle$ or equivalently, $c > 0$. Fig. 2 shows an example of $Cay(S : G)$ under this classification.

The next two propositions describe a hamiltonian decomposition of such graphs if additional conditions are satisfied.

Proposition 2: Let Γ be a degree 4 Cayley graph of type $\Gamma(\alpha, \beta)$ with parameters a, b, α, β, c and k such that $G \neq \langle a \rangle \oplus \langle b \rangle$.

If α and β are both even, let H_1 be the subgraph of Γ defined by the following sequence:

$$0, \left(\frac{k-\beta-c}{c}\right) * [\alpha * (+b)],\ +b,\ \left(\frac{\alpha}{2} - 1\right) * [+b,\ \beta * (+a),\ +b,\ \beta * (-a)], +b,$$
$$\left(\frac{(c-\beta)(k-\beta)}{\beta c}\right) * [\alpha * (+b)],\ +b,$$
$$(\beta - 1) * (-a),\ -b,\ \left(\frac{k}{\beta} - 1\right) * [\alpha * (-b)],$$
$$+a,\ \left(\frac{\beta}{2} - 1\right) * \left[\left(\frac{k}{\beta} - 1\right) * [\alpha * (+b)],\right.$$
$$\left. +a,\ \left(\frac{k}{\beta} - 1\right) * [\alpha * (-b)],\ +a\right],$$

and let $H_2 = \Gamma \setminus E(H_1)$. Then H_1 and H_2 form a hamiltonian decomposition of Γ.

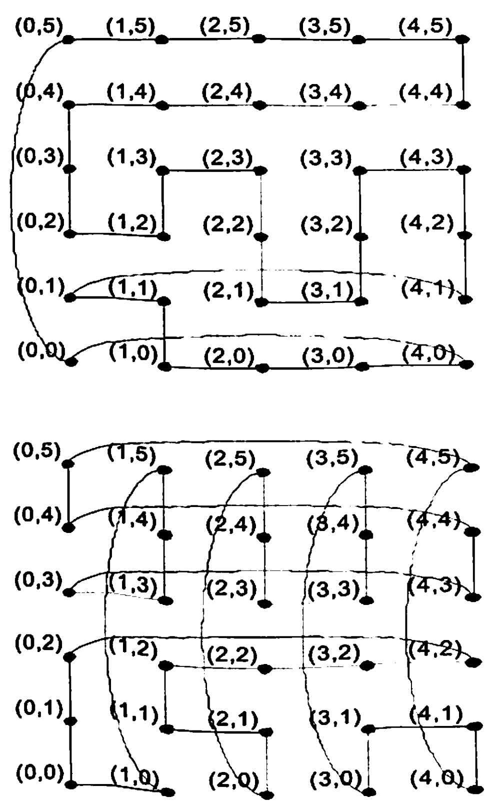

Figure 7. Hamiltonian decomposition of $Cay(\{5, 12\} : \mathbb{Z}_{30})$.

To prove Proposition 2, we note that a hamiltonian decomposition of Γ that has type $\Gamma(\alpha, \beta)$ can be obtained from a graph of type $\Gamma(2, \beta)$ with the same parameters c and k, according to Lemma 1. Now a graph of type $\Gamma(2, \beta)$ also has type $\Gamma(\beta, 2)$ which can be decomposed from a decomposition of a graph of type $\Gamma(2, 2)$, again according to Lemma 1. Finally this graph of type $\Gamma(2, 2)$ has a hamiltonian decomposition described in Lemma 2.

Figure 8 illustrates the decomposition described in the previous proposition for the Cayley graph $Cay(\{a, b\} : G)$ where $G = \mathbb{Z}_2 \oplus \mathbb{Z}_{20}$, $a = (1, 2)$ and $b = (0, -3)$.

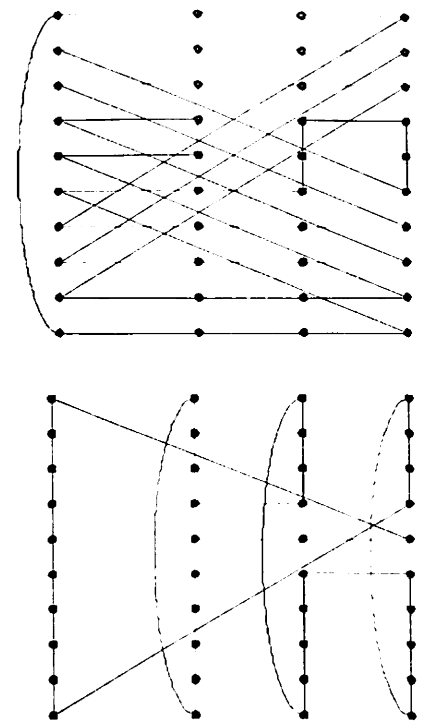

Figure 8. Hamiltonian decomposition of $Cay(\{(1,2), (0,-3)\} : \mathbb{Z}_2 \oplus \mathbb{Z}_{20})$.

Proposition 3: Let Γ be a Cayley graph with parameters a, b, α, β, c and k such that $G \neq \langle a \rangle \oplus \langle b \rangle$.

If α is odd, β is even and $c \neq k/2$, let H_1 be the subgraph of Γ defined by the following sequence:

$$0, \left(\frac{k-\beta-c}{c}\right) * [\alpha * (+b)],\ +b,\ \left(\frac{\alpha-1}{2}\right) * [+b,\ \beta * (+a),\ +b,\ \beta * (-a)], +b,$$
$$\left(\frac{(c-\beta)(k-\beta)}{\beta c}\right) * [\alpha * (+b)],$$
$$(\beta - 1) * (-a),\ \left(\frac{\alpha-1}{2}\right) * [-b,$$
$$(\beta - 2) * (+a),\ -b,\ (\beta - 2) * (-a)],$$
$$-b,\ \left(\frac{k}{\beta} - 2\right) * [\alpha * (-b)],\ +a,$$
$$\left(\frac{\beta}{2} - 1\right) * \left[\left(\frac{k}{\beta} - 2\right) * [\alpha * (+b)],\right.$$
$$\left. +a,\ \left(\frac{k}{\beta} - 2\right) * [\alpha * (-b)],\ +a\right],$$

and let $H_2 = \Gamma \setminus E(H_1)$. Then H_1 and H_2 form a hamiltonian decomposition of Γ.

Like the previous two results, Proposition 3 can be proved using Lemma 1 and Lemma 2. In the process, note that the assumption $c \neq k/2$ is necessary to obtain a valid four regular graph of a particular type.

Figure 9 illustrates the decomposition for the graph of $Cay(\{5, 4\} : \mathbb{Z}_{60})$.

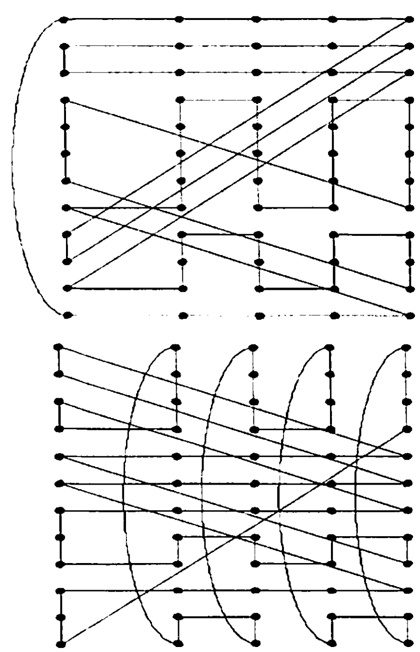

Figure 9. Hamiltonian decomposition of $Cay(\{5, 4\} : \mathbb{Z}_{60})$.

Conclusion

Depending on certain conditions, we have constructed hamiltonian decompositions for certain classes of degree four Cayley graphs on abelian groups. In particular we obtained an algorithm to decompose $Cay(S : G)$ when it has type $\Gamma(\alpha, \beta)$ for all cases of α and β (and the parameters c and k), except the following:

- α and β are both odd and $c \neq 0$, or
- one of α and β is even and the other odd and $c = k/2$.

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